CONIC SECTIONS AND THEIR APPLICATIONS

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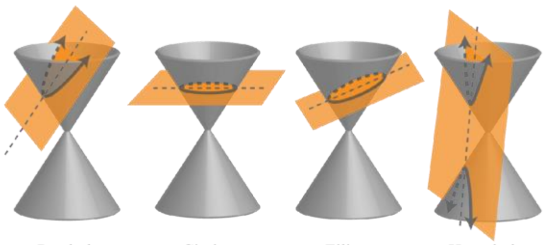
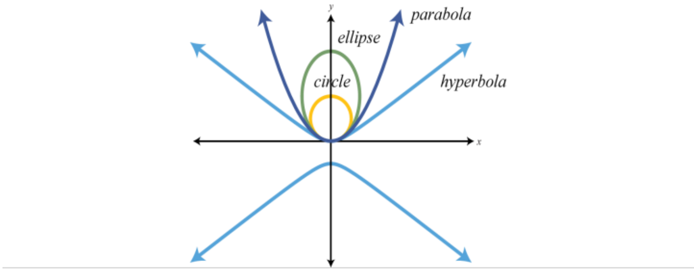
Abstract. *Teachers' mastery of subject areas and ability to effectively Impact the knowledge on the students are the major rudiments expected from an effective teacher. Conic sections coordinate geometry are regarded by students as difficult. Conic sections are a very powerful conceptual framework for bringing algebra, geometry, history of Mathematics, applications and use in many other fields of knowledge together. It is thus, a rich point of departure for the idea of integration or interconnectedness in and around Mathematics which is making the learning of Mathematics more meaningful and further higher-order thinking skills. This study presents Conic sections and their applications in different areas were discussed.*
Keywords: *Conic sections, circle, hyperbola, ellipse, parabola.*

If a straight line and a point be given in position in a plane, and if a point moves in a plane in such a manner that its distance from the given point always bears the same ratio to its distance from the given line, the curve traced out by the moving point is called a Conic Section. The fixed point is called the Focus, and the fixed line the Directrix of the conic section. When the ratio is one of equality, the curve is called a Parabola. When the ratio is one of less inequality, the curve is called an Ellipse. When the ratio is one of greater inequality, the curve is called a Hyperbola. These curves are called Conic Sections, because they can all be obtained from the intersections of a Cone by planes in different directions, a fact which will be proved hereafter. It may be mentioned that a circle is a particular case of an ellipse, that two straight lines constitute a particular case of an hyperbola, and that a parabola may be looked upon as the limiting form of an ellipse or an hyperbola, under certain conditions of variation in the lines and magnitudes upon which those curves depend for their form. The object of the following pages is to discuss the general forms and characters of these curves, and to determine their most important properties by help of the methods and relations developed in the first six books, and in the eleventh book of Euclid, and it will be found that, for this purpose, a knowledge of Euclid's Geometry is all that is necessary. The series of demonstrations will shew the characters and properties which the curves possess in common, and also the special characteristics wherein they differ from each other; and the continuity with which the curves pass into each other will appear from the definition of a conic section as a Locus, or curve traced out by a moving point, as well as from the fact that they are deducible from the intersections of a cone by a succession of planes.

Definition. a conic section is a curve obtained from the intersection of a right circular cone and a plane. The conic sections are the parabola, circle, ellipse, and hyperbola.



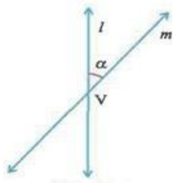


The goal is to sketch these graphs on a rectangular coordinate plane.

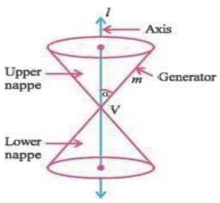
1. Formation of Conics

Conics are formed by intersection of a Double Cone and a Plane

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and let the measure of the angle made by m with l be $\alpha (0 < \alpha < \pi/2)$. Suppose the line m is rotated around the line l in such a way that the angle α remains constant.

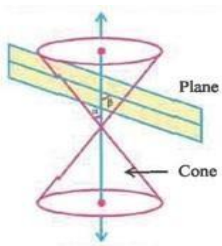


Then the surface generated is called a right circular cone. The point of intersection V separates the cone in two parts. Hence it is called a double napped cone or a double cone. For simplicity we will refer this as a cone. Since the lines l and m are of infinite extent. The cone is extending indefinitely in both the directions. The point V is called the vertex. The line l is the axis of the cone and the rotating line m is called a generator of the cone, and two parts of the cone are called napes.



The intersection of a plane with a cone, the section so obtained is called a conic section. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane and hence the name conics.

There are many possibilities when we consider intersection of a cone with a plane depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let $\beta (0 < \beta < \pi/2)$. be the angle made by the plane with the vertical axis of the cone.



There are two possibilities:

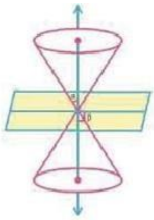
1) The plane passes through the vertex.

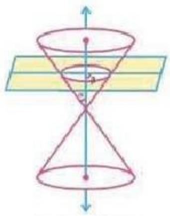
2) The plane does not pass through the vertex.

Accordingly, the intersection takes place at vertex or at any other part of the napes above or below the vertex.

Various situation of intersection is discussed below; in each case above two possibilities are discussed separately.

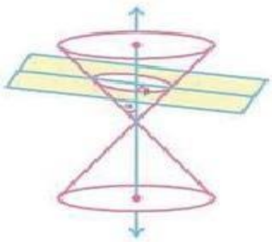
Let the angle made by the plane with the axis of the cone be right angle, i.e. $\beta = {}^{\pi}/_{2}$. If the plane passing through the vertex, then the intersection is the vertex itself;



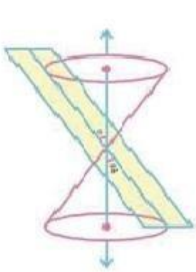


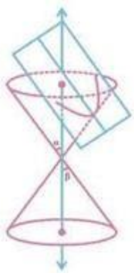
and if the plane does not pass through the vertex, then the intersection is the **circle**, either in the upper nape of the cone or the lower nape of the cone depending on the positions of the plane. In the first case we got the intersection as a point. Thus it is a degenerate case of the circle.

Suppose $\alpha < \beta < {}^{\pi}/_{2}$. If the plane is passing through the vertex, then the intersection is the vertex itself. If it is not the case, then the intersection is the **ellipse**. Here also the first case is degenerate ellipse – a point.

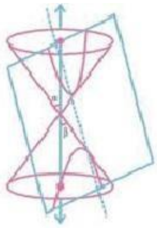


Now, consider the case, when $\alpha = \beta$, in this case intersecting plane is parallel to a generator. If the plane passes through the vertex, then the intersection is a straight line. It can be seen that the line of intersection is a generator of a cone. If the vertex is not on the plane, then the intersection is a **parabola**. The intersection being a straight line is actually degenerate parabola, i.e. as if parabola is opened up straight to get the line.





Finally consider the case $\alpha > \beta$. In this case the plane intersects both the napes. This did not happen in earlier cases. The intersection is a **hyperbola** and it has 2 branches as shown in Fig. Here the degeneracy occurs in a particular case in which plane passes through the vertex and the intersection is a pair of lines.



We have seen that circle, ellipse, parabola and hyperbola are various conics with point, line or a pair of lines as degenerate cases.

2. Applications of Conics

Circles

Circles are defined as a set of points that are equidistant (the same distance) from a certain point; this distance is called the **radius** of a circle.

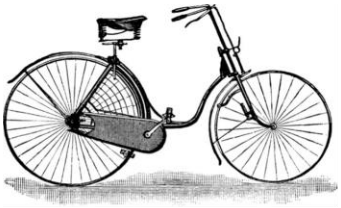
Here is the equation for a circle, where r is the radius.

$$center(0,0): x^2 + y^2 = r^2$$
$$center\ (h,k): (x-h)^2 + (y-k)^2 = r^2$$

3. Applications of Circles:

- Wheels

Circles are the best shape for a bicycle because they roll very easily as they are round. The center point would be the (h,k) in the equation and all points along the outer edge would be the (x,y)values. The radius would be represented by the bars supporting the wheel that run from center to the outer rim.



- Clock

It is a cycle of 60 seconds, 60 min and 12 hours. 60/12 = 5 hence 5 minutes' increments, circle because cycles are circular they repeat once the cycle runs through easiest way to repeat the cycle in a circular or loop.

Parabola:

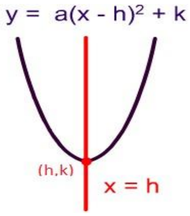
A parabola is in the form $y = a(x-h)^2 + k$, where (h,k) is

the **vertex** and "***x* = *h***" is the **axis of symmetry** or **line of symmetry**

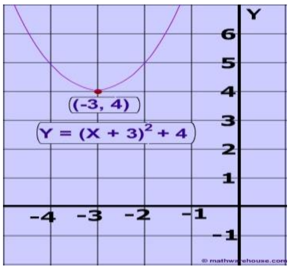
(LOS); this is a "vertical" parabola. It can also be in the form $x = a(y-h)^2 + k$, where (h,k) is the **vertex**, and "***y* = *k***" is the **LOS.**

The **line of symmetry (LOS)** is a line that divides the parabola into two parts that are mirror images of each other.

Technically, a parabola is the set of points that are equidistant from a line (called the **directrix**) and another point not on that line (called the **focus,** or **focal point**).



Example: What is the vertex of $y = (x+3)^2 + 4$.
Answer: $(3, 4)$.

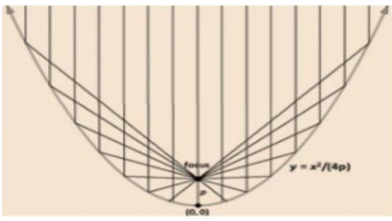


4. Applications of Parabola:

- Parabolic mirror

A parabolic mirror at the Seattle Science Museum

Question: The equation $\boldsymbol{y = \frac{x^2}{32}}$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of its parabola?



Assuming the vertex of the parabola is at (0,0). Since we know that the equation of parabola $y = ax^2$, where $a = \frac{1}{4p}$. Comparing it with $y = \frac{x^2}{32}$ we get $a = \frac{1}{32}$ and p=8. So the heating tube needs to be located 8 units above the vertex of the parabola.

- Parabolic receivers:

A person who whispers at the focus of one of the parabolic reflectors can be heard by a person located near the focus of the other parabola.

- Satellite Dish:

The shape of a cross section of a satellite antenna is a parabola. The shape of the antenna is a paraboloid. The 'dish' part of the antenna is just a reflective surface. The actual antenna is the object held up in the center by an arm that comes off the side of the dish. The antenna is positioned at the focus. Any energy that is parallel to the axis of the parabola will reflect back to the focus of the parabola regardless of where the energy strikes the surface.

- Path of Water fountain:

A projectile (such as a drop of water) ejected from the fountain has an initial velocity V_0 which has components V_{x0} and V_{y0} which are horizontal and vertical components of the initial velocity. If we ignore air resistance, thenV_{x0} is actually the horizontal velocity throughout the flight of the drop. So we'll just call it V_x.

However, V_y does change. It starts at V_{y0} upwards (let's call upwards positive) and when the drop reaches the top of the arc, it has reduced to 0. Then it increases in magnitude, but becomes negative, until it strikes the pool of water at the same speed downwards as it started with upwards. (assuming here that the jet of water is at the same elevation as the pool into which it falls.)

The equation of projectile motion that describes the vertical distance (height) of the drop is just

$$y = V_{y0}t + \left({}^{1}\!/_{2}\right)gt^2,$$

where t. is the time in seconds and g is the gravitational acceleration (negative) in distance per second.

The equation of the projectile's horizontal motion is uncelebrated, so it is just $x = V_{xt}$.

Assuming that the starting point where the drop is ejected from the jet of water is the origin $y_0 = 0$ and $x_0 = 0$

Now we have two equations that describe what x and y are for a given initial velocity with respect to time. This is what we call a parameterized graph.

So let's see if this is a parabola. If it is, we should be able to write it without reference to t, but with known quantities involved in the form $y = ax^2 + bx + c$, where a, b and c are just scalar quantities based on known initial conditions (i.e. the initial velocity components).

First, if $x = V_{xt}$, then $t = {}^{x}/_{V_x}$

So since $y = V_{y0}t + \left({}^{1}/_{2}\right)gt^2$, we can substitute for t with the expression above.

$y = V_{y0}\left({}^{x}/_{V_x}\right) + \left({}^{1}/_{2}\right)g\left({}^{x}/_{V_x}\right)^2$ which we can reorganize as:

$y = \left[{}^{g}/_{(2V_x)^2}\right]x^2 + \left[{}^{V_{y0}}/_{V_x}\right]x + [0]$

Since this is indeed in the form of a parabola, this shows that the fountain arc is indeed parabolic.

Ellipse

5. Lithotripsy – A Medical Application of the Ellipse

The ellipse is a very special and practical conic section. One important property of the ellipse is its reflective property.

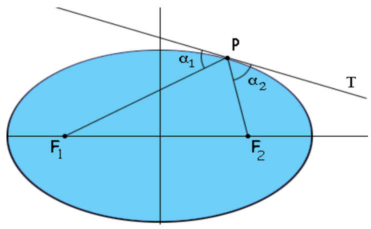
If we think of an ellipse as being made from a reflective material, then a light ray emitted from one focus will reflect off the ellipse and pass through the second focus.

This is also true not only for light rays, but also for other forms of energy, including shockwaves.

Shockwaves generated at one focus will reflect off the ellipse and pass through the second focus. This characteristic, unique to the ellipse, has inspired a useful medical application.

Medical specialists have used the ellipse to create a device that effectively treats kidney stones and gallstones.

A *lithotripter* uses shockwaves to successfully shatter a painful kidney stone (or gallstone) into tiny pieces that can be easily passed by the body. This process is known as lithotripsy.



As illustrated in the diagram above, when as energy ray reflects off a surface, the angle of incidence is equal to the angle of reflection.

$\alpha_1 = \alpha_2$

Extracorporeal Shockwave Lithotripsy (ESWL) enables doctors to treat kidney and gall stones without open surgery.

By using this alternative, risks associated with surgery are significantly reduced. There is a smaller possibility of infections and less recovery time is required than for a surgical procedure.

The lithotripter is the instrument used in lithotripsy.

The mathematical properties of an ellipse provide the basis for this medical invention.

The Foci

The lithotripter machine has a half ellipsoid shaped piece that rests opening to the patient’s side.

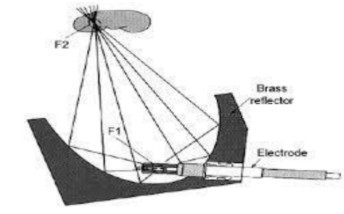


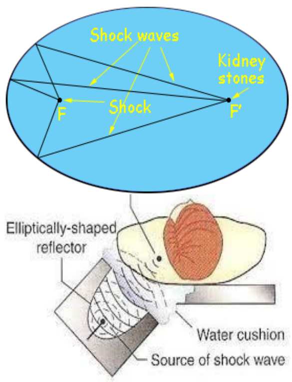
Figure 1 – Shockwave focus system.

An ellipsoid is a three dimensional representation of an ellipse. In order for the lithotripter to work using the reflective property of the ellipse, the patient's stone must be at one focus point of the ellipsoid and the shock-wave generator at the other focus.

The patient is laid on the table and moved into position next to the lithotripter.

Doctors use a fluoroscopic x-ray system to maintain a visual of the stone. This allows for accurate positioning of the stone as a focus. Because the stone is acting as one of the focus points, it is imperative that the stone be at precisely the right distance from the focus located on the lithotripter.

This is essential in order for the shock-waves to be directed onto the stone.



The Cushion

The lithotripter also contains a coupling device. This is needed for the successful transmission of the rays through the body. A cushion, somewhat like a water balloon, wraps around the half ellipsoid. The cushion is filled with water and rests against the patient's side. The cushion is sealed to the patient's body using a silicone membrane. It is the water that allows the shockwaves to travel through the body's tissues safely because water and the soft tissue have the same density. The stone has a greater density and is shattered by the shockwaves, but the soft tissue suffers only minimal damage. Before the new lithotripters were made, patients would lay in a water bath to create the same effect.

Shockwaves

Electrohydraulic, piezoelectric, and electromagnetic energy systems use the focus of the ellipsoid to create the shockwaves needed to fracture the stone. The waves are generated at one focus and because of the elliptical shape, the waves are redirected onto the second focus, which is the stone itself. All of these waves cause the stone to crack and it eventually fragments into many tiny pieces that can then be easily passed by the body.


Image of lithotripter

The process of lithotripsy takes about an hour. The patient can usually return home the same day and is not subjected to a lengthy recovery that is frequently required after surgery. Lithotripsy is virtually painless. The vibration and noise of the shocks can be uncomfortable and so most patients require minimal anesthesia.

The lithotripter reflects shock waves into the patient's kidneys to break up kidney stones into fragments small enough to pass via their urine. For these reasons, lithotripsy is becoming a popular treatment for many patients.

2. Optics

Ellipses have important applications to optics.

Ellipses show up in nearly all fields of optics. The two foci of an ellipse allow opticians to accurately predict the path of a beam of light. By manipulating the angle and size of an elliptical lens, they can magnify, refract or reflect light. They then use these properties to create microscopes, telescopes and cameras. Physicists and engineers use the optical properties of ellipses to determine how much light scatters and how much an object absorbs -- two important properties of laser mechanics.

NASA

Without the ability to actually go to planets and take physical measurements, astrophysicists have to make estimates about their measurements. In 2003, NASA sent remote-controlled land vehicles to Mars. To give the astronauts a visual landing zone they used the shape of an ellipse. The ellipse gave them an area inside of safe terrain and the mathematics of ellipses allowed them to better calculate the chance of landing outside of the predicted zone.

6. Application Hyperbola:
- Cooling Towers of Nuclear Reactors

The hyperboloid is the design standard for all nuclear cooling towers. It is structurally sound and can be built with straight steel beams.

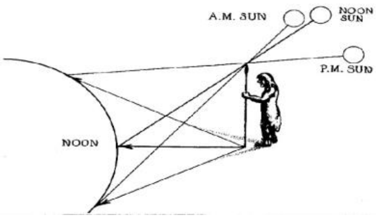
When designing these cooling towers, engineers are faced with two problems: (1) the structure must be able to withstand high winds and (2) they should be built with as little material as possible.

The hyperbolic form solves both of these problems. For a given diameter and height of a tower and a given strength, this shape requires less material than any other form. A 500-foot tower can be made of a reinforced concrete shell only six or eight inches wide. See the pictures below (this nuclear power plant is located in Indiana).

- Sundials

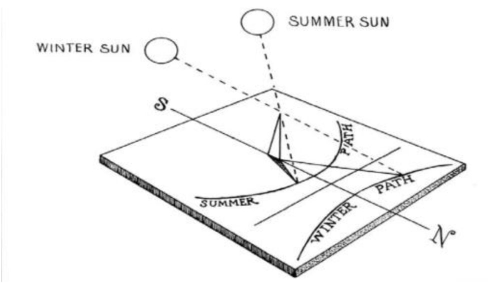
A sundial, in its broadest sense, is any device that uses the motion of the apparent sun to cause a shadow or a spot of light to fall on a reference scale indicating the passage of time. The invention of the sundial is lost in the obscurity of ancient times. But we can imagine some of the factors that led up to its invention.

Very early in our history, humans must have observed the shadows cast by trees and noticed that the shadows grew shorter as the morning wore on and then grew longer again after midday. Thus, if a shadow a little after sunup was twice as long as the height of a stick, then, in the afternoon, when the shadow was again twice as long as the height of the stick, there would be that same amount of time left before sundown.



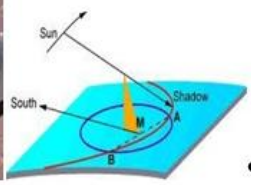
The shadow of a vertical object like the spear, or in later times a tall obelisk, can be used in two ways to indicate the passing of time. The first method, which uses the changing length of the shadow, is illustrated above. The second method uses the changing direction of the shadow. In the morning as the sun rises in the east, the shadow points west. Then, as the day advances, the shadow first swings to the north and then to the east, where it points when the sun sets in the west.

The direction method has been preferred historically over the length of shadow method. The problem with the latter is that shadows are shorter in the summer than in the winter because the earth is tilted toward the sun in summer and away from the sun in the winter.



The shortest shadow is traced about June 21, and the longest about December 22. The straight-line shadow occurs about March 21 and again about September 23. Such a basic knowledge of the seasons was essential to the earliest agricultural societies, so we might assume that their need gave rise to the first primitive sundials. The Greek historian Herodotus (484-425 B.C.) stated in his writings that the sundial originated in Babylonia in the fertile valleys of the Tigris and Euphrates rivers.

Hyperbolas may be seen in many sundials. Every day, the sun revolves in a circle on the celestial sphere, and its rays striking the point on a sundial traces out a cone of light. The intersection of this cone with the horizontal plane of the ground forms a conic section. At most populated latitudes and at most times of the year, this conic section is a hyperbola. The shadow of the tip of a pole traces out a hyperbola on the ground over the course of a day (this path is called the declination line). The shape of this hyperbola varies with the geographical latitude and with the time of the year, since those factors affect the cone of the sun's ray's relative to the horizon.



Conclusion.

Ability of students to understand difficult concepts and solve problems in Mathematics highly depends on the interest of such student's right from elemental. Also they have interest to the application of some different areas in mathematics. We have attempted to show in this study "conic sections and their applications" before students learn some knowledge they should be given a problem. This will be help and encourage students to learn conic sections easily. For more clarity we brought some favorite usage of the conic sections.

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КОНИЧЕСКИЕ СЕЧЕНИЯ И ИХ ПРИМЕНЕНИЕ

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Аннотация. *Владение преподавателями предметными областями и способность эффективно воздействовать на знания учащихся являются основными зачатками, ожидаемыми от эффективного преподавателя. Геометрия координат конических сечений рассматривается студентами как сложная. Конические разделы являются очень мощной концептуальной основой для объединения алгебры, геометрии, истории математики, приложений и использования во многих других областях знаний. Таким образом, это богатая отправная точка для идеи интеграции или взаимосвязанности в математике и вокруг нее, которая делает изучение математики более значимым и развивает навыки мышления более высокого порядка. В этом исследовании представлены конические сечения, и были обсуждены их приложения в различных областях.*

Ключевые слова: *конические сечении, круг, гипербола, парабола, эллипс.*